

Real differential Galois groups

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Abstract

For a linear differential equation defined over a real differential field K with real closed field of constants, we determine the number of K -differential isomorphism classes of Picard-Vessiot extensions and the differential Galois group for each of them, in the case of simple groups.

1 Introduction

In this paper we consider linear differential equations defined over a real differential field with real closed field of constants. In the sequel, K will always denote a real differential field with field of constants k and we shall assume that k is real closed.

In [4], we proved the existence and unicity up to K -differential isomorphism of a real Picard-Vessiot extension for a linear differential equation $\mathcal{L}(Y) = 0$ defined over the real differential field K . We note that such a linear differential equation may also have non real Picard-Vessiot extensions. Let L be a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$. Then $G = \mathrm{DGal}(L|K)$ is a linear algebraic group defined over the real closed field of constants k . It is known that the set of K -differential isomorphism classes of Picard-Vessiot extensions for $\mathcal{L}(Y) = 0$ is in one-to-one correspondence with the cohomology set $H^1(k, G(\bar{k}))$, where \bar{k} denotes the algebraic closure of k . For a $\mathrm{Gal}(\bar{k}|k)$ -module A , we use the notation $H^n(k, A)$ for $H^n(\mathrm{Gal}(\bar{k}|k), A)$.

In [4] Proposition 3.3, we proved that when the differential field K is real closed, given a connected semi-simple linear algebraic group G defined over k , there exists a linear differential equation defined over K and a real Picard-Vessiot extension $L|K$ for it such that $G = \mathrm{DGal}(L|K)$.

Let now G denote a linear algebraic group defined over a real closed field k and let $H = G \times_k \bar{k}$. The set of forms of H over k is in one-to-one correspondence with the cohomology set $H^1(k, \mathrm{Aut} G(\bar{k}))$. We consider the map

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$$\Phi : H^1(k, G(\bar{k})) \rightarrow H^1(k, \text{Aut } G(\bar{k}))$$

induced by the morphism from $G(\bar{k})$ to $\text{Aut } G(\bar{k})$ sending an element g in $G(\bar{k})$ to conjugation by g . When G is the differential Galois group of a Picard-Vessiot extension L of K for a linear differential equation $\mathcal{L}(Y) = 0$, Φ sends the element in $H^1(k, G(\bar{k}))$ corresponding to a Picard-Vessiot extension L_1 of K for $\mathcal{L}(Y) = 0$ to the element in $H^1(k, \text{Aut } G(\bar{k}))$ corresponding to $\text{DGal}(L_1|K)$ (see [4] 3.1).

For a linear differential equation defined over a real differential field K the determination of its real differential group G gives more information on the behavior of the solutions than the determination of the complexification H of G . It is interesting to study how does the real differential group $\text{DGal}(L|K)$ vary as L runs over the K -isomorphism classes of Picard-Vessiot extensions. In this paper we use the classification of simple linear algebraic groups over a real closed field (see [2], [9]) to determine the differential Galois groups of the different K -differential isomorphism classes of Picard-Vessiot extensions of $\mathcal{L}(Y) = 0$, in the cases when G is simple.

We refer the reader to [3] for the topics on differential Galois theory, to [1] for those on real fields.

2 Preliminaries

Let $L|K$ be a real Picard-Vessiot extension for a linear differential equation $\mathcal{L}(Y) = 0$ defined over a real field K , with differential Galois group a simple linear differential group G defined over the real closed field of constants k of K . We want to determine the differential Galois group of each of the Picard-Vessiot extensions of K for $\mathcal{L}(Y) = 0$.

The Galois group $\text{Gal}(\bar{k}|k)$ acts on $G(\bar{k})$, by an involution leaving G invariant, and on $\text{Aut } G(\bar{k})$ by $s(f) = s \circ f \circ s^{-1}$, for $s \in \text{Gal}(\bar{k}|k)$, $f \in \text{Aut } G(\bar{k})$, as usual. Let us note that, even if the action of $\text{Gal}(\bar{k}|k)$ on $\text{Aut } G(\bar{k})$ depends on the chosen form G of $G \times_k \bar{k}$, the cohomology set $H^1(k, \text{Aut } G(\bar{k}))$ does not.

Let us denote by c the non trivial element of $\text{Gal}(\bar{k}|k)$ and write $\bar{a} = c(a)$, for a an element in \bar{k} . For $v = (a_1, \dots, a_n) \in \bar{k}^n$, we shall write $\bar{v} = (\bar{a}_1, \dots, \bar{a}_n)$ and for $M = (a_{ij})$ a matrix with entries in \bar{k} , $\bar{M} = (\bar{a}_{ij})$.

We recall that a simple linear algebraic group defined over an algebraically closed field is isomorphic to a member of one of the following infinite families

$\mathrm{SL}(n)$, special linear group,
 $\mathrm{SO}(n)$, special orthogonal group,
 $\mathrm{Sp}(n)$, symplectic group,

or to one of the exceptional groups

$$G_2, F_4, E_6, E_7, E_8.$$

We will analyze the real forms of each of these groups. In the case of the special orthogonal group, one needs to distinguish the cases n odd and n even.

In the sequel, K will denote a real differential field, k its constant field, which we assume to be real closed, $\mathcal{L}(Y) = 0$ a linear differential equation defined over K , $L|K$ a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$, which is a k -defined linear algebraic group. We shall assume that G is simple and will determine in each case the number of K -differential isomorphism classes of Picard-Vessiot extensions of K for $\mathcal{L}(Y) = 0$ and the differential Galois group for each class.

Let i denote a square root of -1 in \bar{k} . For p, n integers with $0 \leq p \leq n$, we define the $n \times n$ matrices

$$I_p = \begin{pmatrix} Id_p & 0 \\ 0 & -Id_{n-p} \end{pmatrix}, \quad J_p = \begin{pmatrix} Id_p & 0 \\ 0 & iId_{n-p} \end{pmatrix}.$$

3 Forms of $\mathrm{SL}(n)$

The real forms of $\mathrm{SL}(n)$, $n \geq 2$, are

1. $\mathrm{SL}(n, k)$;
2. $\mathrm{SL}(n/2, \mathbb{H})$ if n is even, where \mathbb{H} denotes the quaternion algebra over k ;
3. $\mathrm{SU}(n, \bar{k}, h)$, where h is a non degenerate hermitian form on \bar{k}^n .

3.1 $G = \mathrm{SL}(n, k)$

We have $|H^1(k, G(\bar{k}))| = 1$ ([7] X §1), hence $L|K$ is the unique Picard-Vessiot extension for $\mathcal{L}(Y) = 0$, up to K -differential isomorphism.

3.2 $G = \mathrm{SL}(n/2, \mathbb{H})$

We shall prove $|H^1(k, G(\bar{k}))| = 2$, by following the same steps as for the corollary to Proposition 3 in [7] X §1. Let us denote by $1, I, J, K$ the basis elements of \mathbb{H} . We recall that $\mathrm{GL}(n/2, \mathbb{H})$ embeds into $\mathrm{GL}(n, \bar{k})$ through the morphism $(h_{ij}) \mapsto (\mu(h_{ij}))$, where

$$\mu(a + bI + cJ + dK) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}, a, b, c, d \in k.$$

We denote by A_n the matrix $(a_{ij})_{1 \leq i, j \leq n}$ with

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is odd and } j = i + 1, \\ -1 & \text{if } i \text{ is even and } j = i - 1, \\ 0 & \text{in all other cases.} \end{cases}$$

We have $\mu(\mathrm{GL}(n/2, \mathbb{H})) = \{M \in \mathrm{GL}(n, \bar{k}) : M = A_n \bar{M} A_n^{-1}\}$ and $\mu(\mathrm{SL}(n/2, \mathbb{H})) = \mathrm{SL}(n, \bar{k}) \cap \mu(\mathrm{GL}(n/2, \mathbb{H}))$. We consider the action of $\mathrm{Gal}(\bar{k}|k)$ on $\mathrm{GL}(n, \bar{k})$ given by $c(M) = A_n \bar{M} A_n^{-1}$. The sequence

$$1 \rightarrow \mathrm{SL}(n, \bar{k}) \longrightarrow \mathrm{GL}(n, \bar{k}) \xrightarrow{\det} \bar{k}^* \rightarrow 1$$

is an exact sequence of $\mathrm{Gal}(\bar{k}|k)$ -modules. With the $\mathrm{Gal}(\bar{k}|k)$ -module structure we are considering on $\mathrm{GL}(n, \bar{k})$ the proof that $H^1(k, \mathrm{GL}(n, \bar{k}))$ is trivial remains valid (see [7] X §1) and we obtain the exact sequence

$$H^0(k, \mathrm{GL}(n, \bar{k})) \xrightarrow{\det} k^* \xrightarrow{\delta} H^1(k, \mathrm{SL}(n, \bar{k})) \rightarrow 1.$$

We compute the image of $\det : H^0(k, \mathrm{GL}(n, \bar{k})) \rightarrow k^*$. For $r \in k$, we have $rId \in H^0(k, \mathrm{GL}(n, \bar{k})) = \{M \in \mathrm{GL}(n, \bar{k}) : c(M) = M\}$ and $\det(rId) = r^n$, hence the image of \det contains all positive elements in k . On the other hand, if $M = c(M)$ and v is an eigenvector of M with eigenvalue λ , then $A_n^{-1} \bar{v}$ is an eigenvector of M with eigenvalue $\bar{\lambda}$, hence $\det M > 0$. We have then $H^1(k, \mathrm{SL}(n, \bar{k})) \simeq \mathrm{Coker}(\det) = k^*/k_+^*$, hence $|H^1(k, G(\bar{k}))| = 2$. A non trivial 1-cocycle x of $\mathrm{Gal}(\bar{k}|k)$ in $\mathrm{SL}(n, \bar{k})$ is given by $x(c) = \zeta Id$, for ζ a primitive n -th root of unity. We have two Picard-Vessiot extensions for $\mathcal{L}(Y) = 0$, up to K -differential isomorphism. A K -differential automorphism of $L(i)$ corresponding to x is given by the matrix $\zeta^{-1/2} Id$ on the vector space of solutions. Conjugation by $\zeta^{-1/2} Id$ leaves the group $\mathrm{SL}(n/2, \mathbb{H})$ stable. We obtain that the Picard-Vessiot extensions for $\mathcal{L}(Y) = 0$ in both K -differential isomorphy classes have $\mathrm{SL}(n/2, \mathbb{H})$ as differential Galois group.

3.3 $G = \mathrm{SU}(n, \bar{k}, h)$

It is known that if h is a non degenerate hermitian form on \bar{k}^n , then h is equivalent to a hermitian form with matrix I_p , for some integer p with $0 \leq p \leq n$, called the index of h and that two non degenerate hermitian forms on \bar{k}^n are equivalent if and only if their indexes coincide.

We fix $G = \{M \in \mathrm{SL}(n, \bar{k}) : \bar{M}^t I_p M = I_p\}$ and consider the action of $\mathrm{Gal}(\bar{k}|k)$ on $\mathrm{SL}(n, \bar{k})$ given by $c(M) = I_p(\bar{M}^t)^{-1} I_p$. We shall prove

$$|H^1(k, G(\bar{k}))| = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{when } n \text{ is odd or } p \text{ is even,} \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{when } n \text{ is even and } p \text{ is odd.} \end{cases}$$

A 1-cocycle from $\mathrm{Gal}(\bar{k}|k)$ in $\mathrm{SL}(n, \bar{k})$ is given by a matrix $B \in \mathrm{SL}(n, \bar{k})$ satisfying $Bc(B) = Id$. We denote by x_q the cocycle given by $c \mapsto B_q$, where $B_q = I_q I_p$, with q an integer of the same parity as p and $0 \leq q \leq n$.

We have $x_q \sim x_{q'} \Leftrightarrow \exists M \in \mathrm{SL}(n, \bar{k})$ such that $B_{q'} = M^{-1} B_q c(M)$. This equality is equivalent to $M I_{q'} \bar{M}^t = I_q$ which implies $q = q'$, so the 1-cocycles x_q are pairwise non equivalent. Let us see now that every 1-cocycle x from $\mathrm{Gal}(\bar{k}|k)$ in $\mathrm{SL}(n, \bar{k})$ is equivalent to some x_q . Such a cocycle x is determined by $x(c) = B$ satisfying $Bc(B) = Id$, i.e. $B I_p = I_p \bar{B}^t = (\bar{B} I_p)^t$, so $B I_p$ is an hermitian matrix, hence there exists an invertible matrix M such that $M^{-1} B I_p (\bar{M}^t)^{-1} = I_r$, for some integer r . Equivalently

$$M^{-1} B c(M) = I_r I_p. \quad (1)$$

Since $B \in \mathrm{SL}(n, \bar{k})$, we have $\det(I_r I_p) = 1$, so x is equivalent to some x_q . Let us note that, taking determinants in (1), we obtain $\det M \det \bar{M} = 1$, hence there exists $\zeta \in \bar{k}$ such that $\det(\zeta M) = 1$ and ζM satisfies (1).

We have $Z(\mathrm{SL}(n, \bar{k})) = \mu_n(\bar{k})$. We want to determine the image of $[x_q]$ under the map $\varphi : H^1(k, G(\bar{k})) \rightarrow H^1(k, \mathrm{Aut} G(\bar{k}))$. The 1-cocycle x_q corresponds to a Picard-Vessiot extension of K for $\mathcal{L}(Y) = 0$ which is determined by an automorphism f_q of $L(i)$ satisfying $x_q = f_q^{-1} \bar{f}_q$. We may take f_q with matrix $D_q := J_q J_p$ over the vector space of solutions. Since conjugation by D_q gives the identity on G , we obtain that all Picard-Vessiot extensions of K for $\mathcal{L}(Y) = 0$ have the same differential Galois group G .

Gathering the results in this section we may state the following theorem.

Theorem 1. *Let K be a real differential field with real closed field of constants k , $\mathcal{L}(Y) = 0$ a linear differential equation defined over K , $L|K$ a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of $\mathrm{SL}(n)$.*

- (1) *If $G = \mathrm{SL}(n, k)$, $L|K$ is the unique Picard-Vessiot extension for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism;*
- (2) *if $G = \mathrm{SL}(n/2, \mathbb{H})$, there are two Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism, and both of them have differential Galois group G ;*
- (3) *if $G = \mathrm{SU}(n, \bar{k}, h)$, there are $[n/2] + 1$ (resp. $[n/2]$) Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, if n is odd or p is even (resp. if n is even and p is odd), up to K -differential isomorphism, and all of them have differential Galois group G .*

4 Forms of $\mathrm{SO}(n)$, n odd

This case has already been treated in [4]. We include it here for the sake of completeness. The real forms of $\mathrm{SO}(n)$, with n odd, are the groups $\mathrm{SO}(n, k, Q)$, where Q is a non degenerate quadratic form on k^n . The quadratic form Q is equivalent to a quadratic form with matrix I_p , for some integer p with $0 \leq p \leq n$, called the index of Q , and two non degenerate quadratic forms on k^n are equivalent if and only if their indexes coincide.

Write $G = \mathrm{SO}(n, k, Q)$, for Q a non degenerate quadratic form on k^n with index p . The cohomology set $H^1(k, G(\bar{k}))$ is in one-to-one correspondence with the set of equivalence classes of quadratic forms on k^n of rank n and discriminant equal to the discriminant of Q , i.e. of index of the same parity as p . We have then $|H^1(k, G(\bar{k}))| = \frac{n+1}{2}$.

The cocycles x_q defined by $c \mapsto B_q$, where $B_q = I_q I_p$, with q an integer with the same parity as p and $0 \leq q \leq n$, form a complete system of representatives of the cohomology set $H^1(k, G(\bar{k}))$. As in 3.3 we may take the automorphism f_q of $L(i)$ corresponding to x_q to have matrix $D_q := J_{2q} J_{2p}$ over the vector space of solutions of $\mathcal{L}(Y) = 0$. If the matrix M satisfies $M^t I_p M = I_p$, the conjugate matrix $N := D_q M D_q^{-1}$ satisfies $N^t I_q N = I_q$, hence the Picard-Vessiot extension corresponding to the 1-cocycle x_q has

differential Galois group $\mathrm{SO}(n, k, Q_q)$, where Q_q denotes the quadratic form with index q .

The results in this section are stated in the following theorem.

Theorem 2. *Let K be a real differential field with real closed field of constants k , $\mathcal{L}(Y) = 0$ a linear differential equation defined over K , $L|K$ a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of $\mathrm{SO}(n)$, with n odd. There are $(n + 1)/2$ Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism, and their differential Galois groups range over the whole set of real forms of $\mathrm{SO}(n)$.*

5 Forms of $\mathrm{Sp}(2n)$

The real forms of $\mathrm{Sp}(2n)$ are

1. $\mathrm{Sp}(2n, k)$;
2. $\mathrm{SU}(n, \mathbb{H}, h)$, where h is a non degenerate hermitian form on \mathbb{H}^n (with respect to the involution $\sigma : a + bI + cJ + dK \mapsto a - bI - cJ - dK$).

5.1 $G = \mathrm{Sp}(2n, k)$

We have $|H^1(k, G(\bar{k}))| = 1$ ([7] X §2, Corollary 2), hence $L|K$ is the unique Picard-Vessiot extension for $\mathcal{L}(Y) = 0$, up to K -differential isomorphism.

5.2 $G = \mathrm{SU}(n, \mathbb{H}, h)$, h hermitian

If h is a non degenerate hermitian form on \mathbb{H}^n , then h is equivalent to a hermitian form with matrix I_p ([2], 7.5.3). We fix

$$G = \{M \in \mathrm{GL}(n, \mathbb{H}) : \sigma(M)^t I_p M = I_p\}.$$

The group G is the group of automorphisms of the hermitian vector space (\mathbb{H}^n, h) . Let us note that for $M, N \in M_n(\mathbb{H})$, we have $\sigma(MN)^t = \sigma(N)^t \sigma(M)^t$. The set of equivalence classes of non degenerate hermitian forms over \mathbb{H}^n is in one-to-one correspondence with the cohomology set $H^1(k, G(\bar{k}))$. We have $\mathrm{GL}(n, \mathbb{H} \otimes_k \bar{k}) \simeq \mathrm{GL}(2n, \bar{k})$, through the morphism $\mu \otimes_k \bar{k}$. For $M \in$

$\mathrm{GL}(n, \mathbb{H} \otimes_k \bar{k})$, we have $(\mu \otimes_k \bar{k})(\sigma(M^t)) = A_n(\mu \otimes_k \bar{k})(M)^t A_n^{-1}$. Hence $\sigma(M)^t I_p M = I_p \Rightarrow (\mu \otimes_k \bar{k})(M)^t A_n^{-1} I_{2p} (\mu \otimes_k \bar{k})(M) = A_n^{-1} I_p$. Now the group

$$\overline{G} := \{N \in \mathrm{GL}(2n, \bar{k}) : N^t A_n^{-1} I_{2p} N = A_n^{-1} I_{2p}\}$$

is a conjugate form of $\mathrm{Sp}(2n, \bar{k})$ and $\mu(G) = \{N \in \overline{G} : N = A_n \overline{N} A_n^{-1}\}$. We have obtained that the number of K -differential isomorphic classes of Picard-Vessiot extensions for $\mathcal{L}(Y) = 0$ is equal to the number of equivalence classes of non degenerate hermitian forms over \mathbb{H}^n , which is $n + 1$. A complete set of non equivalent 1-cocycles is given by

$$\begin{array}{ccc} x_q : \mathrm{Gal}(\bar{k}|k) & \rightarrow & \overline{G} \\ c & \mapsto & B_q := I_{2q} I_{2p} \end{array},$$

with q an integer, $0 \leq q \leq n$. We may take the automorphism f_q of $L(i)$ corresponding to x_q to have matrix $D_{2q} := J_{2q} J_{2p}$ over the vector space of solutions of $\mathcal{L}(Y) = 0$. If the matrix N satisfies $N^t A_n^{-1} I_{2p} N = A_n^{-1} I_{2p}$, the conjugate matrix $P := D_{2q} N D_{2q}^{-1}$ satisfies $P^t A_n^{-1} I_{2q} P = A_n^{-1} I_{2q}$, hence the Picard-Vessiot extension corresponding to the 1-cocycle x_q has differential Galois group $\mathrm{SU}(n, \mathbb{H}, h_q)$, where h_q denotes the hermitian form with index q .

Gathering the results in this section we may state the following theorem.

Theorem 3. *Let K be a real differential field with real closed field of constants k , $\mathcal{L}(Y) = 0$ a linear differential equation defined over K , $L|K$ a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of $\mathrm{Sp}(2n)$.*

- (1) *If $G = \mathrm{Sp}(2n, k)$, $L|K$ is the unique Picard-Vessiot extension for the equation $\mathcal{L}(Y) = 0$;*
- (2) *if $G = \mathrm{SU}(n, \mathbb{H}, h_p)$, where h_p is a non degenerate hermitian form on \mathbb{H}^n , of index p , $0 \leq p \leq n$, there are $n + 1$ Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism, and their differential Galois groups range over the whole set of groups $G = \mathrm{SU}(n, \mathbb{H}, h_q)$, with h_q a non degenerate hermitian form on \mathbb{H}^n , of index q , $0 \leq q \leq n$.*

6 Forms of $\mathrm{SO}(n)$, n even

The real forms of $\mathrm{SO}(n)$, with n even, are

1. $\mathrm{SO}(n, k, Q)$, where Q is a non degenerate quadratic form on k^n ;
2. $\mathrm{SU}(n/2, \mathbb{H}, h)$, where h is a non degenerate anti-hermitian form on $\mathbb{H}^{n/2}$ (with respect to the involution σ).

6.1 $G = \mathrm{SO}(n, k, Q)$

The quadratic form Q is equivalent to a quadratic form with matrix I_p , for some integer p with $0 \leq p \leq n$, which determines the equivalence class of Q .

The cohomology set $H^1(k, G(\bar{k}))$ is in one-to-one correspondence with the set of equivalence classes of quadratic forms on k^n of rank n and index of the same parity as p . We have then

$$|H^1(k, G(\bar{k}))| = \begin{cases} \frac{n}{2} + 1 & \text{when } p \text{ is even,} \\ \frac{n}{2} & \text{when } p \text{ is odd.} \end{cases}$$

The cocycles x_q defined by $c \mapsto B_q$, where $B_q = I_q I_p$, with q an integer of the same parity as p and $0 \leq q \leq n$, form a complete system of representatives of the cohomology set $H^1(k, G(\bar{k}))$. We may take the automorphism f_q of $L(i)$ corresponding to x_q to have matrix $D_q := J_q J_p$ over the vector space of solutions of $\mathcal{L}(Y) = 0$. If the matrix M satisfies $M^t I_p M = I_p$, the conjugate matrix $N := D_q M D_q^{-1}$ satisfies $N^t I_q N = I_q$, hence the Picard-Vessiot extension corresponding to the 1-cocycle x_q has differential Galois group $\mathrm{SO}(n, k, Q_q)$, where Q_q denotes the quadratic form with index q . Let us note that $\mathrm{SO}(n, k, Q_q) = \mathrm{SO}(n, k, Q_{n-q})$, hence the Picard-Vessiot extension corresponding to x_q and x_{n-q} , $0 \leq q \leq (n/2) - 1$, have the same differential Galois group.

6.2 $G = \mathrm{SU}(n/2, \mathbb{H}, h)$, h anti-hermitian

We have $G = \{M \in \mathrm{GL}(n/2, \mathbb{H}) : \sigma(M)^t [h] M = [h]\}$, for $[h]$ the matrix of the anti-hermitian form h , in some basis of $\mathbb{H}^{n/2}$. The group G is the group of automorphisms of the anti-hermitian vector space $(\mathbb{H}^{n/2}, h)$. The set of equivalence classes of non degenerate hermitian forms over $\mathbb{H}^{n/2}$ is in one-to-one correspondence with the cohomology set $H^1(k, G(\bar{k}))$. Up to equivalence, there is one single anti-hermitian form on \mathbb{H}^n , hence $H^1(k, G(\bar{k})) = 1$. We may check that $\mu(G)$ is the intersection of (a conjugate form of) $\mathrm{SO}(n, \bar{k})$ with

$\mu(\mathrm{GL}(n/2, \mathbb{H}))$. We have obtained that $L|K$ is the unique Picard-Vessiot extension for $\mathcal{L}(Y) = 0$, up to K -differential isomorphism.

Gathering the results in this section we may state the following theorem.

Theorem 4. *Let K be a real differential field with real closed field of constants k , $\mathcal{L}(Y) = 0$ a linear differential equation defined over K , $L|K$ a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of $\mathrm{SO}(n)$, with n even.*

- (1) *If $G = \mathrm{SO}(n, k, Q_p)$, where Q_p is a non degenerate quadratic form on k^n , of index p , there are $(n/2) + 1$ (resp. $n/2$) Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism, when p is even (resp. when p is odd) and their differential Galois groups range over the whole set of groups $G = \mathrm{SO}(n, k, Q_q)$, with Q_q a non degenerate quadratic form on k^n , of index q , $0 \leq q \leq n/2$ and q of the same parity as p ;*
- (2) *if $G = \mathrm{SU}(n/2, \mathbb{H}, h)$, where h is a non degenerate anti-hermitian form on \mathbb{H}^n , $L|K$ is the unique Picard-Vessiot extension for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism.*

7 Forms of the exceptional groups

In this section we restrict our study to the groups G_2, F_4 and E_8 . For $G = G_2, F_4$ or E_8 , we have $G \simeq \mathrm{Aut} G$, which makes easier our study of Picard-Vessiot extensions and their Galois differential groups. For E_6 and E_7 not all real forms are explicitly described. For E_6 , there are 5 real forms, of which 2 are inner. The simply connected form of E_6 can be seen as the group of transformations of an Albert algebra which leave a certain cubic form invariant. For E_7 , there are 4 real forms, which are all inner. The simply connected form of E_7 can be seen as the identity component of the group of transformations of an Albert algebra leaving invariant a certain quartic form. The center of this form of E_7 has order 2 (cf. [8] 17.7, 17.8).

7.1 G_2

The compact form of the group G_2 over a real closed field k is the group of automorphisms of the division algebra \mathbb{O} of the octonions. Hence the

cohomology set $H^1(k, G_2(\bar{k}))$ is in one-to-one correspondence with the set of k -isomorphism classes of k -algebras which are \bar{k} -isomorphic to \mathbb{O} . In turn, this set is in one-to-one correspondence with the set of equivalence classes of 3-Pfister forms over k (see [8] 17.4). A 3-Pfister form over a field k is the quadratic form obtained as the tensor product

$$\langle 1, a \rangle \otimes \langle 1, b \rangle \otimes \langle 1, c \rangle = \langle 1, a, b, ab, c, ac, bc, abc \rangle,$$

with $a, b, c \in k \setminus \{0\}$. It is easily checked that there are two equivalence classes of 3-Pfister forms over k and we obtain $|H^1(k, G_2(\bar{k}))| = 2$. Hence a linear differential equation $\mathcal{L}(Y) = 0$ defined over K , with differential Galois group G_2 , has two Picard-Vessiot extensions, up to K -differential isomorphism.

We have $\text{Aut } G_2 \simeq G_2$, hence $H^1(k, G_2(\bar{k}))$ classifies also the real forms of G_2 . The second form (the split form) is the group G'_2 of automorphisms of the split octonion algebra. Since $\text{Aut } G_2 \simeq G_2$, the map $\Phi : H^1(k, G_2(\bar{k})) \rightarrow H^1(k, \text{Aut } G_2(\bar{k}))$ is trivially bijective and so, if $L|K$ is a real Picard-Vessiot extension for a linear differential equation $\mathcal{L}(Y) = 0$ defined over K , with differential Galois group G_2 (resp. G'_2), then the non real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over K has differential Galois group G'_2 (resp. G_2).

7.2 F_4

A linear algebraic group of type F_4 is the group of automorphisms of a so-called Albert k -algebra \mathbf{A} (see [8] 17.5). We have $|H^1(k, F_4(\bar{k}))| = 3$ and $\text{Aut } F_4 \simeq F_4$, whence $H^1(k, F_4(\bar{k}))$ classifies the real forms of F_4 up to k -isomorphism. A linear differential equation $\mathcal{L}(Y) = 0$ defined over K , with differential Galois group F_4 , has three Picard-Vessiot extensions, up to K -differential isomorphism, and each of the three real forms of F_4 appears as the differential Galois group of one of the Picard-Vessiot extensions.

7.3 E_8

The group E_8 has three real forms, which are not easily described. We have $\text{Aut } E_8 \simeq E_8$, hence a linear differential equation $\mathcal{L}(Y) = 0$ defined over K , with differential Galois group E_8 , has three Picard-Vessiot extensions, up to K -differential isomorphism, and each of the three real forms of E_8 appears as the differential Galois group of one of the Picard-Vessiot extensions.

Gathering the results in this section we may state the following theorem.

Theorem 5. *Let K be a real differential field with real closed field of constants k , $\mathcal{L}(Y) = 0$ a linear differential equation defined over K , $L|K$ a real Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ and G the differential Galois group of $L|K$. We assume that G is a real form of one of the exceptional groups G_2, F_4 or E_8 .*

- (1) *If G is a real form of G_2 , there are 2 Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism;*
- (2) *If G is a real form of F_4 or E_8 , there are 3 Picard-Vessiot extensions for the equation $\mathcal{L}(Y) = 0$, up to K -differential isomorphism.*

In all three cases, the differential Galois groups of the Picard-Vessiot extensions range over the whole set of real forms of $G \times_k \bar{k}$.

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